

Second Semester Examination 2009-10

Mathematics-II

Section-A

Note : Attempt all questions. Each question carries equal marks.

(10 × 2 = 20)

Q. 1. (a) The order and degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^4 - 6x^2\left(\frac{dy}{dx}\right)^8 = 0$ are

..... and

Ans. Order = 3, Degree = 4

Q. 1. (b) Pick the correct statement from the following :

(i) Integrating factor to a differential equation is unique.

(ii) $y = ex$ is the general solution of the $\frac{ydx - xdy}{x^2} = 0$.

Ans. (ii) $y = ex$ is the general solution of the $\frac{ydx - xdy}{x^2} = 0$.

Q. 1. (c) Write the Rodrigue formula $P_n(x) = \dots$

Ans. Rodrigue formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$

Q. 1. (d) If J_0 and J_1 are Bessel's functions, then $J_1(x)$ is given by :

(i) $J_0(x) - \frac{1}{x} J_1(x)$

(ii) $J_0 + \frac{1}{x} J_1(x)$

(iii) $-J_0(x)$

(iv) None of these

Ans. (iv) None of these

Q. 1. (e) $L^{-1}\left\{\frac{1}{S^n}\right\}$ exist only when the value of n is

(i) Negative integer

(ii) Positive integer

(iii) Zero

(iv) None of these

Ans. (ii) Positive integer

Q. 1. (f) Pick the correct statement for final value theorem of Laplace transform :

(i) $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow \infty} sF(s)$

(ii) $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

Ans. (ii) $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

Q. 1. (g) Fourier coefficient ' a_0 ' in Fourier series expansion of a function represents the :

(i) maximum value of the function

(ii) mean value of the function

(iii) minimum value of the function

(iv) none of these

Ans. (ii) Mean value of the function.

Q. 1. (h) If the Fourier series of $f(x)$ has only cosine terms then $f(x)$ must be :

(i) odd function

(ii) even function

Q. 1. (i) The PDE $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ is known as :

- (i) wave equation (ii) heat equation
(iii) Laplace equation (iv) none of these

Ans. (i) Wave equation

Q. 1. (j) The PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, is :

- (i) parabolic (ii) elliptic
(iii) hyperbolic (iv) circular

Ans. (ii) Elliptic

Section-B

Note : Attempt any three parts from this section. Each part carry equal marks : (3 × 10 = 30)

Q. 2. (a) A particle of mass m moves in a straight line under the action of force mn^2x which is always directed towards a fixed point O on the line. Determine the displacement $x(t)$ if the resistance to the motion is $2\lambda mnv$ given that initially $x = 0$, $\frac{dx}{dt} = x_0$ ($0 < \lambda < 1$).

Ans. The differential equation describing this simple harmonic motion is

$$m \frac{d^2x}{dt^2} = -2\lambda mnv - mn^2x, \Rightarrow m \frac{d^2x}{dt^2} = -2\lambda mn \frac{dx}{dt} - mn^2x$$

$$\frac{d^2x}{dt^2} + 2\lambda n \frac{dx}{dt} + n^2x = 0 \quad \dots(1)$$

The auxiliary equation is $M^2 + 2\lambda nM + n^2 = 0$

$$M = \frac{-2\lambda n \pm \sqrt{4\lambda^2 n^2 - 4n^2}}{2} = -\lambda n \pm in\sqrt{1 - \lambda^2}$$

The general solution is $x = e^{-\lambda nt} [c_1 \cos n\omega t + c_2 \sin n\omega t]$... (2)

where $\omega = \sqrt{1 - \lambda^2}$,

using I.C. $x = 0, t = 0 \Rightarrow c_1 = 0$

Differentiating (2), we get

$$\frac{dx}{dt} = e^{-\lambda nt} [-c_1 n \omega \sin n\omega t + c_2 n \omega \cos n\omega t] - \lambda n e^{-\lambda nt} [c_1 \cos n\omega t + c_2 \sin n\omega t]$$

Again using I.C. $\frac{dx}{dt} = x_0, t = 0$

$$x_0 = -c_1 \cdot 0 + c_2 n \omega - \lambda n [c_1 + c_2 \cdot 0] \quad c_2 = \frac{x_0}{n\omega}$$

From (2), its solution is $x = e^{-\lambda nt} \left(\frac{x_0}{n\omega} \right) \sin n\omega t$

Q. 2. (b) Using Frobenius method obtain a series solution in powers of x for differential equation : $2x(1-x) \frac{d^2y}{dx^2} + (1-x) \frac{dy}{dx} + 2y = 0$

Ans. $2x(1-x) \frac{d^2y}{dx^2} + (1-x) \frac{dy}{dx} + 3y = 0$... (1)

$\therefore x=0$ is a regular singular point

Let $y = x^m \sum_{n=0}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_n x^{m+n}$... (2)

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2}$$

Substituting these values in equation (1), we get

$$2x[1-x] \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2} + (1-x) \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1} + 3 \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\begin{aligned} \sum_{n=0}^{\infty} a_n [2(m+n)(m+n-1) + (m+n)] x^{m+n-1} \\ + \sum_{n=0}^{\infty} a_n [-2(m+n)(m+n-1) - (m+n) + 3] x^{m+n} = 0 \\ \sum_{m=0}^{\infty} a_n (m+n)(2m+2n-1) x^{m+n-1} - \sum_{n=0}^{\infty} a_n (m+n+1)(2m+2n-3) x^{m+n} = 0 \dots (3) \end{aligned}$$

Coefficient of $x^{m-1} = 0$

$$a_0(m)(2m-1) = 0 \Rightarrow a_0 \neq 0$$

$$m(2m-1) = 0 \Rightarrow$$

$$m = 0, \frac{1}{2}$$

It is the root of indicial equation.

Coefficient of $x^m = 0$

$$a_1(m+1)(2m+1) - a_0(m+1)(2m-3) = 0 \Rightarrow a_1 = \left(\frac{2m-3}{2m+1} \right) a_0$$

Coefficient of $x^{m+n} = 0$

$$a_{n+1}(m+n+1)(2m+2n+1) - a_n(m+n+1)(2m+2n-3) = 0$$

$$a_{n+1} = \left(\frac{2m+2n-3}{2m+2n+1} \right) a_n \dots (4)$$

$$n=1, \quad a_2 = \left(\frac{2m-1}{2m+3} \right) a_1 = \left(\frac{2m-1}{2m+1} \right) \left(\frac{2m-3}{2m+3} \right) a_0$$

$$n=2, \quad a_3 = \left(\frac{2m+1}{2m+5} \right) a_2 = \left(\frac{2m-1}{2m+3} \right) \left(\frac{2m-3}{2m+5} \right) a_0$$

Substituting these values of a_1, a_2, a_3 in equation (2) we get

$$y = x^m \left[1 + \left(\frac{2m-3}{2m+1} \right) x + \left(\frac{2m-1}{2m+3} \right) \left(\frac{2m-3}{2m+1} \right) x^2 + \left(\frac{2m-1}{2m+5} \right) \left(\frac{2m-3}{2m+1} \right) \left(\frac{2m-3}{2m+3} \right) x^3 + \dots \right]$$

Now, $y_1 = (y)_{m=0}$

$$y = a_0 x^0 \left[1 - 3x + \frac{3}{3} x^2 + \frac{3}{15} x^3 \dots \right] = a_0 \left[1 - 3x + x^2 + \frac{x^3}{5} + \dots \right] \quad \dots(6)$$

$$y_2 = (y)_{m=1/2} = a_0 x^{1/2} \left[1 - \frac{2}{2} x \right] = a_0 x^{1/2} (1 - x)$$

Hence, the complete solution is, $y = c_1 y_1 + c_2 y_2$

$$= A \left(1 - 3x + x^2 + \frac{x^3}{5} \dots \right) + B x^{1/2} (1 - x)$$

where $A = c_1 a_0$ and $B = c_2 a_0$.

Q. 2. (c) Using Laplace transform, solve the differential equation :

$$\frac{d^2 y}{dx^2} + n^2 y = a \sin (nx + 2)$$

Given, $y(0) = 0$ and $y'(0) = 0$.

Ans. $\frac{d^2 y}{dx^2} + n^2 y = a \sin (nx + 2)$, Given $y(0) = 0$, $y'(0) = 0$

$$y'' + n^2 y = a (\sin nx \cos 2 + \cos nx \sin 2)$$

Taking Laplace transform both sides, we get

$$L(y'') + n^2 L(y) = a \cos 2 L(\sin nx) + a \sin 2 L(\cos nx)$$

$$s^2 L(y) - y(0) - y'(0) + n^2 L(y) = a \cos 2 \left(\frac{n}{s^2 + n^2} \right) + a \sin 2 \left(\frac{s}{s^2 + n^2} \right)$$

$$(s^2 + n^2) L(y) = a \cos 2 \left(\frac{n}{s^2 + n^2} \right) + a \sin 2 \left(\frac{s}{s^2 + n^2} \right)$$

$$L(y) = a \cos 2 \left[\frac{n}{(s^2 + n^2)^2} \right] + a \sin 2 \left[\frac{s}{(s^2 + n^2)^2} \right]$$

Taking inverse Laplace transform both sides,

$$y = a \cos 2 L^{-1} \left\{ \frac{n}{(s^2 + n^2)^2} \right\} + a \sin 2 L^{-1} \left\{ \frac{s}{(s^2 + n^2)^2} \right\} \quad \dots(1)$$

$$L^{-1} \left\{ \frac{s}{(s^2 + n^2)^2} \right\} = \frac{x}{2n} \sin nx$$

$$L^{-1} \left\{ \frac{n}{(s^2 + n^2)^2} \right\} = \frac{1}{2n^2} (\sin nx - nx \cos nx)$$

$$y = a \cos 2 \left[\frac{1}{2n^2} (\sin nx - nx \cos nx) \right] + a \sin 2 \left(\frac{x}{2n} \sin nx \right)$$

Q. 2. (d) Find the Fourier series to represent the function $f(x)$ given by :

$$f(x) = \begin{cases} -K & \text{for } -\pi < x < 0 \\ K & \text{for } 0 < x < \pi \end{cases}$$

Hence show that : $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$.

Ans. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$... (1)

$$\begin{aligned} \text{Where } a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 (-K) dx + \int_0^{\pi} (K) dx \right] \\ &= \frac{1}{\pi} [-K(x)_{-\pi}^0 + K(x)_0^{\pi}] = \frac{1}{\pi} [-K(-\pi - 0) + K(\pi - 0)] = \frac{1}{\pi} [K\pi + K\pi] = 2K \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \left[\int_{-\pi}^0 (-K) \cos nx dx + \int_0^{\pi} K \cos nx dx \right] \\ &= \frac{1}{\pi} \left[-K \left(\frac{\sin nx}{n} \right)_{-\pi}^0 + K \left(\frac{\sin nx}{n} \right)_0^{\pi} \right] = 0 \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \left[\int_{-\pi}^0 (-K) \sin nx dx + \int_0^{\pi} K \sin nx dx \right] \\ &= \frac{1}{\pi} \left[(-K) \left(\frac{-\cos nx}{n} \right)_{-\pi}^0 + K \left(\frac{-\cos nx}{n} \right)_0^{\pi} \right] = \frac{1}{\pi} \left[+\frac{K}{n} (1 - \cos n\pi) - \frac{K}{n} (\cos n\pi - 1) \right] \\ &= \frac{K}{n\pi} [2(1 - \cos n\pi)] = \frac{2K}{n\pi} [1 - (-1)^n] \end{aligned}$$

The Fourier series to the function $f(x)$

$$\begin{aligned} f(x) &= \frac{2K}{2} - \frac{2K}{\pi} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n} \sin nx \\ f(x) &= K - \frac{2K}{\pi} \left[\frac{2 \sin x}{1} + \frac{2 \sin 3x}{3} + \frac{2 \sin 5x}{5} + \dots \right] \\ f(x) &= K - \frac{4K}{\pi} \left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right] \end{aligned}$$

put $x = \frac{\pi}{2}$

$$K = K \left[1 - \frac{4}{\pi} \left(\frac{\sin \frac{\pi}{2}}{1} + \frac{\sin \frac{3\pi}{2}}{3} + \frac{\sin \frac{5\pi}{2}}{5} + \dots \right) \right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Q. 2. (e) Solve the Laplace's equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

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in a rectangle in the xy -plane, $0 \leq x \leq a$ and $0 \leq y \leq b$ satisfying the following boundary

Ans. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$... (1)

Let $u = X(x) Y(y)$... (2)

where X is function of x only and Y is function of y only.

(1) $\Rightarrow X'' Y + X Y'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant } (+p^2)$$

$$\frac{X''}{X} = +p^2 \Rightarrow \frac{d^2 X}{dx^2} - p^2 X = 0$$

Auxiliary equation is $m^2 - p^2 = 0 \Rightarrow m = \pm p$

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$-\frac{Y''}{Y} = p^2 \Rightarrow \frac{d^2 Y}{dy^2} + p^2 Y = 0$$

Auxiliary equation is $m^2 + p^2 = 0 \Rightarrow m = \pm pi$

$$Y = c_3 \cos py + c_4 \sin py$$

$\therefore u(x, y) = (c_1 e^{px} + c_2 e^{-px}) (c_3 \cos py + c_4 \sin py)$... (3)

Using B.C., $u(x, 0) = 0 \Rightarrow (c_1 e^{px} + c_2 e^{-px}) c_3 = 0 \Rightarrow c_3 = 0$... (4)

Using B.C., $u(x, b) = 0 \Rightarrow (c_1 e^{px} + c_2 e^{-px}) (c_4 \sin pb) = 0 \Rightarrow c_4 \sin pb = 0$

$$\sin pb = 0 = \sin n\pi \Rightarrow p = \frac{n\pi}{b}$$
 ... (5)

Using B.C., $u(0, y) = 0 \Rightarrow (c_1 + c_2) (c_4 \sin py) = 0 \Rightarrow c_2 = -c_1$... (6)

From equation (3), $u(x, y) = c_1 c_4 \left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}} \right) \sin \frac{n\pi y}{b}$

$$u(x, y) = 2c_1 c_4 \sinh \left(\frac{n\pi x}{b} \right) \sin \frac{n\pi y}{b}$$

The general solution is, $u(x, y) = \sum_{n=1}^{\infty} b_n \sinh \left(\frac{n\pi x}{b} \right) \sin \frac{n\pi y}{b}$... (7)

Using $u(a, y) = f(y)$

$$f(y) = \sum_{n=1}^{\infty} b_n \sinh \left(\frac{n\pi a}{b} \right) \sin \frac{n\pi y}{b},$$

$$\left(\sinh \frac{n\pi a}{b} \right) b_n = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} dy$$

$$b_n = \frac{2}{b} \operatorname{cosech} \left(\frac{n\pi a}{b} \right) \int_0^b f(y) \sin \frac{n\pi y}{b} dy$$
 ... (8)

Hence the required solution is given by (7) and b_n can be determined by (8).

Section-C

Note : Attempt any two parts from all questions of this section. Each part carry equal marks :

Q. 3. (a) Solve the following differential equation : (5 × 2 × 5 = 50)

Ans. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = \sin 3x + \cos 2x$

The auxiliary equation is $m^2 + 5m - 6 = 0$

$$\Rightarrow (m + 6)(m - 1) = 0 \Rightarrow m = -6, 1$$

$$\text{C.F.} = c_1 e^{-6x} + c_2 e^x$$

$$\text{P.I.} = \frac{1}{D^2 + 5D - 6} (\sin 3x + \cos 2x)$$

$$= \frac{1}{D^2 + 5D - 6} \sin 3x + \frac{1}{D^2 + 5D - 6} \cos 2x$$

$$= \frac{1}{-9 + 5D - 6} \sin 3x + \frac{1}{-4 + 5D - 6} \cos 2x$$

$$= \frac{1}{5D - 15} \sin 3x + \frac{1}{5D - 10} \cos 2x$$

$$= \frac{(D + 3)}{5(D^2 - 9)} \sin 3x + \frac{(D + 2)}{5(D^2 - 4)} \cos 2x$$

$$= \frac{(D + 3)}{5(-9 - 9)} \sin 3x + \frac{(D + 2)}{5(-4 - 4)} \cos 2x$$

$$= -\frac{1}{90} [3 \cos 3x + 3 \sin 3x] - \frac{1}{40} [-2 \sin 2x + 2 \cos 2x]$$

$$= -\frac{1}{30} (\cos 3x + \sin 3x) + \frac{1}{20} (\sin 2x - \cos 2x)$$

The complete solution, is $y = \text{C.F.} + \text{P.I.}$

$$= c_1 e^{-6x} + c_2 e^x - \frac{1}{30} (\cos 3x + \sin 3x) + \frac{1}{20} (\sin 2x - \cos 2x)$$

Q. 3. (b) Solve the following: $\frac{dx}{dt} = 3x + 8y$, $\frac{dy}{dt} = -x - 3y$ with $x(0) = 6$ and $y(0) = -2$.

Ans. Writing D for $\frac{d}{dt}$, the given equations become

$$(D - 3)x - 8y = 0 \quad \dots(1)$$

$$x + (D + 3)y = 0 \quad \dots(2)$$

with $x(0) = 6$ and $y(0) = -2$

To eliminate y , operating on both sides of (1) by $(D + 3)$ and on both sides of (2) by 8 and adding, we get

$$[(D - 3)(D + 3) + 8]x = 0$$

$$(D^2 - 1)x = 0$$

Its auxiliary equation is, $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$x = \text{C.F.} = c_1 e^{-t} + c_2 e^t \quad \dots(3)$$

Now, $\frac{dx}{dt} = -c_1 e^{-t} + c_2 e^t$

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$$8y = -4c_1e^{-t} - 2c_2e^t, \quad y = -\frac{c_1}{2}e^{-t} - \frac{c_2}{4}e^t \quad \dots(4)$$

Given, $x(0) = 6$ and $y(0) = -2$,

From (3) and (4)

$$c_1 + c_2 = 6 \quad \text{and} \quad -\frac{c_1}{2} - \frac{c_2}{4} = -2 \Rightarrow c_1 = 2, c_2 = 4$$

$$\therefore \quad x = 2e^{-t} + 4e^t, \quad y = -e^{-t} - e^t$$

Q. 3. (c) Solve: $x \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 - \frac{dy}{dx} = 0.$

Ans. $x \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 - \frac{dy}{dx} = 0 \quad \dots(1)$

put $\frac{dy}{dx} = p \Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx}$

$$(1) \Rightarrow x \frac{dp}{dx} + x p^2 - p = 0, \Rightarrow x \frac{dp}{dx} - p = -xp^2$$

$$\left(-\frac{1}{p^2} \right) \frac{dp}{dx} + \frac{1}{x} \left(\frac{1}{p} \right) = 1 \quad \dots(2)$$

put $\frac{1}{p} = v \Rightarrow -\frac{1}{p^2} \frac{dp}{dx} = \frac{dv}{dx}$

$$(2) \Rightarrow \frac{dv}{dx} + \frac{1}{x} v = 1 \quad \dots(3)$$

I.F. = $e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Multiplying (3) by I.F. and integrating, we get

$$v \cdot x = \frac{x^2}{2} + c_1 \Rightarrow v = \frac{x^2 + 2c_1}{2x} \Rightarrow \frac{1}{p} = \frac{x^2 + 2c_1}{2x}$$

$$p = \frac{2x}{x^2 + 2c_1} \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2 + 2c_1} \Rightarrow dy = \frac{2x}{x^2 + 2c_1} dx$$

Again integrating $y = \log(x^2 + 2c_1) + c_2$

Q. 4. (a) Show that $J'_n(x) = -J_1(x) + \frac{1}{x} J_2(x).$

Ans. We have, $x J'_n = nJ_n - xJ_{n+1} \Rightarrow J'_n = \frac{n}{x} J_n - J_{n+1}$

put $n = 1, \quad J'_1 = \frac{1}{x} J_1 - J_2 \quad \dots(1)$

Differentiating, $J''_1 = \frac{1}{x} J'_1 - \frac{1}{x^2} J_1 - J'_2 \quad \dots(2)$

By recurrence relation, $xJ'_n = nJ_n - xJ_{n+1}$

$$xJ'_{n+1} = -(n+1)J_{n+1} + xJ_n$$

$$J'_2 = -\frac{2}{x} J_2 + J_1 \quad \dots(3)$$

put (1) and (3) in (2)

$$J''_1 = \frac{1}{x} \left(\frac{1}{x} J_1 - J_2 \right) - \frac{1}{x^2} J_1 - \left(-\frac{2}{x} J_2 + J_1 \right) = \frac{1}{x^2} J_1 - \frac{1}{x} J_2 - \frac{1}{x^2} J_1 + \frac{2}{x} J_2 - J_1$$

$$J''_1 = -J_1 + \frac{1}{x} J_2$$

Q. 4. (b) Prove that : $x P_{n-1}^1(x) + n P_{n-1}(x) = P_n^1(x)$.

Ans. $x P'_{n-1}(x) + n P_{n-1}(x) = P'_n(x)$

Recurrence relation $(2n+1) P_n = P'_{n+1} - P'_{n-1}$

put $n = n-1$, $(2n-1) P_{n-1} = P'_n - P'_{n-2}$...(1)

Recurrence relation, $n P_n = x P'_n - P'_{n-1}$
 $n = n-1$

$$(n-1) P_{n-1} = x P'_{n-1} - P'_{n-2} \quad \dots(2)$$

Subtracting (2) from (1), we get

$$(2n-1 - n+1) P_{n-1} = (P'_n - P'_{n-2}) - (x P'_{n-1} - P'_{n-2})$$

$$n P_{n-1} = P'_n - x P'_{n-1}$$

$$x P'_{n-1} + n P_{n-1} = P'_n$$

Q. 4. (c) Evaluate $\int_{-1}^1 x^2 P_n^2(x) dx$.

Ans. $I = \int_{-1}^1 x^2 P_n^2(x) dx$...(1)

$$(2n+1) x P_n = (n+1) P_{n+1} + n P_{n-1}$$

$$x P_n = \frac{1}{2n+1} [(n+1) P_{n+1} + n P_{n-1}]$$

$$I = \int_{-1}^1 \frac{1}{(2n+1)^2} [(n+1) P_{n+1} + n P_{n-1}]^2 dx$$

$$= \frac{1}{(2n+1)^2} \int_{-1}^1 [(n+1)^2 (P_{n+1})^2 + n^2 P_{n-1}^2 + 2n(n+1) P_{n+1} P_{n-1}] dx$$

$$= \frac{1}{(2n+1)^2} \left[(n+1)^2 \int_{-1}^1 (P_{n+1})^2 dx + n^2 \int_{-1}^1 (P_{n-1})^2 dx + 2n(n+1) \int_{-1}^1 P_{n+1} P_{n-1} dx \right]$$

$$= \frac{1}{(2n+1)^2} \left[(n+1)^2 \frac{2}{2(n+1)+1} + n^2 \frac{2}{2(n-1)+1} + 2n(n+1) \cdot 0 \right]$$

$$= \frac{1}{(2n+1)^2} \left[\frac{2(n+1)^2}{(2n+3)} + \frac{2n^2}{(2n-1)} \right]$$

Q. 5. (a) Find the Laplace transform of the following periodic function :

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases} \quad \text{with period is } \frac{2\pi}{\omega}$$

$$\begin{aligned}
 L\{f(t)\} &= \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t) dt \\
 &= \frac{1}{1 - e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0 dt \right] \\
 &= \frac{1}{1 - e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2 + \omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega} \\
 &= \frac{\omega e^{-\pi s/\omega} + \omega}{(1 - e^{-2\pi s/\omega})(s^2 + \omega^2)} = \frac{\omega(1 - e^{-\pi s/\omega})}{(1 - e^{-\pi s/\omega})(1 + e^{-\pi s/\omega})(s^2 + \omega^2)} \\
 &= \left(\frac{\omega}{s^2 + \omega^2} \right) \left(\frac{1}{1 + e^{-\pi s/\omega}} \right)
 \end{aligned}$$

Q. 5. (b) Find the inverse Laplace transform of:

(i) $\frac{s+1}{s^2-6s+25}$

(ii) $\frac{s}{(s^2+4)^2}$

Ans. (i) $L^{-1}\left[\frac{s+1}{s^2-6s+25}\right] = L^{-1}\left[\frac{(s-3)+4}{(s-3)^2+16}\right]$

$$\begin{aligned}
 &= e^{3t} L^{-1}\left[\frac{s+4}{s^2+16}\right] \\
 &= e^{3t} \left[L^{-1}\left(\frac{s}{s^2+16}\right) + L^{-1}\left(\frac{4}{s^2+16}\right) \right] \\
 &= e^{3t} [\cos 4t + \sin 4t]
 \end{aligned}$$

Ans. (ii) Let $f(t) = L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$

$$\begin{aligned}
 \frac{f(t)}{t} &= L^{-1}\left[\int_s^\infty \frac{\dot{s}}{(s^2+4)^2} ds\right] = L^{-1}\left[\left\{-\frac{1}{2(s^2+4)}\right\}_s^\infty\right] \\
 &= L^{-1}\left[\frac{1}{2(s^2+4)}\right] = \frac{1}{2} L^{-1}\left(\frac{1}{s^2+4}\right) \\
 \frac{f(t)}{t} &= \frac{1}{2} \cdot \frac{1}{2} \sin 2t \\
 f(t) &= \frac{t}{4} \sin 2t
 \end{aligned}$$

Q. 5. (c) An alternative emf $E \sin \omega t$ is applied to circuit with an inductance L and a capacitance C in series. Show that the current in the circuit is:

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$$\frac{E}{2} \frac{1}{\sqrt{1-n^2}} (\cos \omega t - \cos nt) \text{ where } n = \frac{1}{LC}$$

$$\begin{aligned}
 L \frac{di}{dt} + \frac{q}{c} &= E \sin \omega t \\
 L \frac{d^2i}{dt^2} + \frac{1}{c} \frac{dq}{dt} &= \omega E \cos \omega t \\
 L \frac{d^2i}{dt^2} + \frac{1}{c} i &= \omega E \cos \omega t \quad \left(\text{Since } i = \frac{dq}{dt} \right) \\
 \frac{d^2i}{dt^2} + \frac{1}{LC} i &= \frac{\omega E}{L} \cos \omega t \\
 \frac{d^2i}{dt^2} + n^2 i &= \frac{\omega E}{L} \cos \omega t \quad \left(\text{Take } n^2 = \frac{1}{LC} \right)
 \end{aligned}$$

Auxiliary equation is,

$$m^2 + n^2 = 0 \Rightarrow m = \pm ni$$

$$\text{C.F.} = c_1 \cos nt + c_2 \sin nt,$$

$$\text{P.I.} = \frac{1}{D^2 + n^2} \frac{\omega E}{L} \cos \omega t$$

$$= \frac{E\omega}{L} \frac{1}{-\omega^2 + n^2} \cos \omega t \quad \text{if } n^2 \neq -\omega^2 = \frac{E\omega}{L(n^2 - \omega^2)} \cos \omega t$$

$$i = \text{C.F.} + \text{P.I.}$$

$$i = c_1 \cos nt + c_2 \sin nt + \frac{E\omega}{L(n^2 - \omega^2)} \cos \omega t \quad \dots(1)$$

$$\frac{di}{dt} = -c_1 n \sin nt + c_2 n \cos nt - \frac{E\omega}{L(n^2 - \omega^2)} \sin \omega t \quad \dots(2)$$

$$\text{Initially } i = 0, \frac{di}{dt} = 0, \text{ when } t = 0$$

$$(1) \Rightarrow, \quad 0 = c_1 + \frac{E\omega}{(n^2 - \omega^2)L}$$

$$c_1 = -\frac{E\omega}{(n^2 - \omega^2)L}$$

$$(2) \Rightarrow, \quad 0 = -c_2 n \Rightarrow c_2 = 0$$

$$\text{From (1) } i = -\frac{E\omega}{(n^2 - \omega^2)L} \cos nt + 0 + \frac{E\omega}{(n^2 - \omega^2)L} \cos \omega t$$

$$i = \frac{E\omega}{(n^2 - \omega^2)L} (\cos \omega t - \cos nt), \text{ where } n^2 = \frac{1}{LC}$$

Q. 6. (a) Solve the partial differential equation $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

Ans. The auxiliary equation are

$$\begin{aligned}
 \frac{dx}{x^2 - yz} &= \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \\
 \frac{x^2 - yz}{dx - dy} &= \frac{y^2 - zx}{dy - dz} = \frac{z^2 - xy}{dz - dx}
 \end{aligned}$$

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$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

On integration, $\log(x - y) = \log(y - z) + \log a$

$$\log\left(\frac{x - y}{y - z}\right) = \log a$$

$$\left(\frac{x - y}{y - z}\right) = a \quad \dots(1)$$

Similarly, taking last two ratio, $\left(\frac{y - z}{z - x}\right) = b \quad \dots(2)$

From (1) and (2), the general solution is $f\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0$

Q. 6. (b) Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$.

Ans. The auxiliary equation is, $m^2 - m = 0 \Rightarrow m(m - 1) = 0$

$$\Rightarrow m = 0, 1$$

$$\text{C.F.} = f_1(y + 0) + f_2(y + x) = f_1(y) + f_2(y + x)$$

$$\text{P.I.} = \frac{1}{D^2 - DD'} (\sin x \cos 2y)$$

$$= \frac{1}{2} \frac{1}{D^2 - DD'} [\sin(x + 2y) + \sin(x - 2y)]$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - DD'} \sin(x + 2y) + \frac{1}{D^2 - DD'} \sin(x - 2y) \right]$$

$$= \frac{1}{2} \left[\frac{1}{-1 + 2} \sin(x + 2y) + \frac{1}{-1 - 2} \sin(x - 2y) \right]$$

$$= \frac{1}{2} \sin(x + 2y) - \frac{1}{6} \sin(x - 2y)$$

The complete solution is

$$z = \text{C.F.} + \text{P.I.}$$

$$= f_1(y) + f_2(y + x) + \frac{1}{2} \sin(x + 2y) - \frac{1}{6} \sin(x - 2y)$$

Q. 6. (c) Solve the partial differential equation $(D + 1)(D + D' - 1)z = \sin(x + 2y)$.

Ans. $(D + 1)(D + D' - 1)z = \sin(x + 2y)$

$$\text{C.F.} = e^{-x} f_1(y) + e^x f_2(y - x)$$

$$\text{P.I.} = \frac{1}{(D + 1)(D + D' - 1)} \sin(x + 2y)$$

$$= \frac{1}{D^2 + DD' + D'} \sin(x + 2y) = \frac{1}{-1 - 2 + D' + 1} \sin(x + 2y)$$

$$= \frac{1}{(D' + 4)} \sin(x + 2y)$$

$$\begin{aligned}
 &= \frac{(D' + 4)}{-4 - 16} \sin(x + 2y) = -\frac{1}{20} [2 \cos(x + 2y) + 4 \sin(x + 2y)] \\
 &= -\frac{1}{10} [\cos(x + 2y) + 2 \sin(x + 2y)]
 \end{aligned}$$

The complete solution is

$$z = \text{C.F.} + \text{P.I.}$$

$$= e^{-x} f_1(y) + e^x f_2(y - x) - \frac{1}{10} [\cos(x + 2y) + 2 \sin(x + 2y)]$$

Q. 7. (a) Solve the following PDE by method of separation of variable

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u, \text{ with } u(0, y) = 0 \text{ and } \frac{\partial u(0, y)}{\partial x} = 1 + e^{-3y}.$$

Ans.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \quad \dots(1)$$

with $u(0, y) = 0, \frac{\partial u}{\partial x}(0, y) = 1 + e^{-3y}$

Let $u = X(x) Y(y)$

where X is a function of x only and Y is a function of y only.

$$(1) \Rightarrow X'' Y = X Y' + 2XY$$

$$\frac{X''}{X} = \frac{Y'}{Y} + 2 = \text{constant} = -p^2$$

$$\frac{X''}{X} = -p^2 \Rightarrow \frac{d^2 X}{dx^2} + p^2 X = 0$$

$$\text{A.E. is, } m^2 + p^2 = 0 \Rightarrow m = \pm pi$$

$$X = c_1 \cos px + c_2 \sin px$$

$$\frac{Y'}{Y} + 2 = -p^2 \Rightarrow \frac{Y'}{Y} = -(p^2 + 2)$$

$$\frac{dY}{Y} = -(p^2 + 2)dy$$

$$\text{Integrating, } \log Y = -(p^2 + 2)y + \log c_3$$

$$Y = c_3 e^{-(p^2 + 2)y}$$

$$u(x, y) = (c_1 \cos px + c_2 \sin px) c_3 e^{-(p^2 + 2)y} \quad \dots(2)$$

$$u(0, y) = 0 \Rightarrow c_1 c_3 e^{-(p^2 + 2)y} = 0 \Rightarrow c_1 c_3 = 0$$

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px) c_3 e^{-(p^2 + 2)y}$$

$$\frac{\partial u}{\partial x}(0, y) = 1 + e^{-3y} \Rightarrow c_2 p c_3 e^{-(p^2 + 2)y} = 1 + e^{-3y}$$

$$\Rightarrow c_2 c_3 p = 1, \quad -(p^2 + 2) = 0$$

and $c_1 c_3 p = 1, \quad -(p^2 + 2) = -3$

$$\Rightarrow c_2 c_3 = \frac{1}{p}, \quad p^2 = -2 \Rightarrow p = i\sqrt{2}$$

$$c_2 c_3 = \frac{1}{i\sqrt{2}}$$

and

$$c_2 c_3 = \frac{1}{p}, \quad p^2 = 1 \Rightarrow p = 1$$

$$c_2 c_3 = 1$$

From (2), the general solution is

$$\begin{aligned} u(x, y) &= \frac{1}{i\sqrt{2}} \sin(i\sqrt{2} \cdot x) + \sin x \cdot e^{-3y} \\ &= \frac{1}{i\sqrt{2}} i \sinh(x\sqrt{2}) + e^{-3y} \sin x \\ &= \frac{1}{\sqrt{2}} \sinh(x\sqrt{2}) + e^{-3y} \sin x \end{aligned}$$

Q. 7. (b) Find the temperature $u(x, t)$ in a slab whose ends $x = 0$ and $x = l$ are kept at temperature zero and whose initial temperature $f(x)$ is given by

$$f(x) = \begin{cases} A & \text{when } 0 < x < \frac{l}{2} \\ 0 & \text{when } \frac{l}{2} < x < l \end{cases}$$

Ans. On dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(1)$$

Boundary conditions are

$$\left. \begin{aligned} u(0, t) &= 0 \\ u(l, t) &= 0 \end{aligned} \right\} \quad \dots(2)$$

Initial condition is

$$u(x, 0) = f(x) = \begin{cases} A, & \text{when } 0 < x < \frac{l}{2} \\ 0, & \text{when } \frac{l}{2} < x < l \end{cases} \quad \dots(3)$$

Let $u = XT$

where X is function of x only and T is function of t only.

From (1) and (2), $XT' = c^2 X''T$

$$\frac{X''}{X} = \frac{T'}{c^2 T} = \text{constant} = -p^2$$

$$\frac{X''}{X} = -p^2 \Rightarrow \frac{d^2 x}{dx^2} + p^2 x = 0$$

A.E. is

$$m^2 + p^2 = 0 \Rightarrow m = \pm pi$$

$$X = c_1 \cos px + c_2 \sin px$$

$$\frac{T'}{c^2 T} = -p^2 \Rightarrow \frac{dT}{T} = -c^2 p^2 dt$$

$$T = c_3 e^{-c^2 p^2 t}$$

$$\text{Hence, } u(x, t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t} \quad \dots(5)$$

using B.C. (2) in (5), we get

$$u(0, t) = 0 \Rightarrow c_1 = 0$$

$$u(l, t) = 0 \Rightarrow c_2 c_3 \sin pl e^{-c^2 p^2 t} = 0$$

$$\sin pl = 0 = \sin n\pi$$

$$p = \frac{n\pi}{l}$$

Hence, from (5),

$$\begin{aligned} u(x, t) &= c_2 c_3 \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c^2 t}{l^2}} \\ &= b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c^2 t}{l^2}}, \text{ where } b_n = c_2 c_3 \end{aligned}$$

The general solution is

$$u(x, t) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c^2 t}{l^2}} \quad \dots(6)$$

using I.C. (3) in (6), we get

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{l} \\ b_n &= \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \\ &= \frac{2}{l} \left[\int_0^{l/2} A \sin \frac{n\pi x}{l} dx + \int_{l/2}^l 0 \cdot \sin \frac{n\pi x}{l} dx \right] \\ &= \frac{2A}{l} \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right)_0^{l/2} \\ &= \frac{2A}{l} \left[-\frac{l}{n\pi} \cos \frac{n\pi}{2} + \frac{l}{n\pi} \right] \\ &= \frac{2A}{n\pi} \left(1 - \cos \frac{n\pi}{2} \right) \end{aligned}$$

Hence, the solution is

$$u(x, t) = \frac{2A}{\pi} \sum_{n=0}^{\infty} \frac{\left(1 - \cos \frac{n\pi}{2} \right)}{n} \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 c^2 t}{l^2}}$$

Q. 7. (c) Solve $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0$ with boundary conditions.

(i) v is finite when $r \rightarrow 0$

(ii) $v = \sum c_n \cos n\theta$ on $r = a$.

Ans. $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0$... (1)

Boundary conditions are

(i) v is finite when $r \rightarrow 0$

(ii) $v = \sum c_n \cos n\theta$ on $r = a$

Let $v = RT$... (2)

where R is a function of r only and T is a function of θ only

$$r^2 R'' T + r R' T + R T'' = 0$$

$$\frac{r^2 R'' + r R'}{R} + \frac{T''}{T} = 0$$

$$\frac{r^2 R'' + r R'}{R} = -\frac{T''}{T} = n^2$$

$$\Rightarrow r^2 R'' + r R' - n^2 R = 0$$

Put $r = e^z \Rightarrow z = \log r, D \equiv \frac{d}{dz}$

$$[D(D-1) + D - n^2] R = 0$$

$$(D^2 - n^2) R = 0$$

A.E. is $m^2 - n^2 = 0 \Rightarrow m = \pm n$

$$R = c_1 e^{nz} + c_2 e^{-nz} = c_1 r^n + c_2 r^{-n}$$

$$\frac{T''}{T} = -n^2 \Rightarrow \frac{d^2 T}{d\theta^2} + n^2 T = 0$$

A.E. is $m^2 + n^2 = 0 \Rightarrow m = \pm ni$

$$T = c_3 \cos n\theta + c_4 \sin n\theta$$

$$v = (c_1 r^n + c_2 r^{-n})(c_3 \cos n\theta + c_4 \sin n\theta)$$

Taking $c_3 = \cos \alpha, c_4 = -\sin \alpha$

$$v = (c_1 r^n + c_2 r^{-n}) \cos(n\theta + \alpha) \quad \dots (3)$$

v is finite when $r \rightarrow 0$, from (3)

$$c_2 = 0$$

(3) $\Rightarrow v = c_1 r^n \cos(n\theta + \alpha) \quad \dots (4)$

$$v = \sum c_n \cos n\theta \text{ on } r = a, \text{ from (4)}$$

$$c_1 = \frac{1}{a^n}, \alpha = 0$$

Hence, from (4) $v = \left(\frac{r}{a}\right)^n \cos n\theta$